Nuclear Criticality Safety Engineer Training
Module 3 [1]

Fission Chain Reactions

LESSON OBJECTIVE

This module introduces the concepts of the infinite multiplication factor, the effective multiplication factor and the components of each.

NEUTRON MULTIPLICATION FACTOR

In a multiplying medium neutrons may be born in fission as high energy neutrons, lose energy by scattering from nuclides in the medium, become thermal neutrons and be captured. Some of the neutrons may be captured in fission events and produce more neutrons, which starts the cycle over again.

The ANSI/ANS Standard 8.1 defines the *effective multiplication factor* (for now, consider only the physical definition given in the standard, not the mathematical definition):

> **effective multiplication factor (k_{eff}).** *Physically*, the ratio of the total number of neutrons produced during a time interval (excluding neutrons produced by sources whose strengths are not a function of fission rate) to the total number of neutrons lost by absorption and leakage during the same interval.

This definition appears to be one that defines something that can be calculated. That is, calculate the number of neutrons produced, the number lost by leakage and absorption, and take a ratio. However, there is a statement about a time interval, which might seem curious for a steady state problem.

Consider a slightly different definition for the *multiplication factor*, k (without the subscript eff for now):

$$k = \frac{\text{number of neutrons in one generation}}{\text{number of neutrons in preceding generation}}$$

[1] Developed for the U. S. Department of Energy Nuclear Criticality Safety Program by T. G. Williamson, Ph.D., Westinghouse Safety Management Solutions, Inc., in conjunction with the DOE Criticality Safety Support Group.

A generation is defined in terms of the neutron history: a neutron is born in fission, slows down, and causes another fission to start the next generation. Here again the concept of time, or generation, is introduced.

Another definition in terms of neutron balance that might be more amenable to computations is:

$$k = P(t) / L(t)$$

$$= \frac{\text{rate of neutron production in a fissionable system}}{\text{rate of neutron loss in a fissionable system}}$$

Neutrons are lost from the fissionable system by absorption or leakage.

Now use the production and loss rates to calculate the time rate of change of the neutron population. Define the *neutron lifetime* as

$$\ell = N(t) / L(t),$$

where N(t), the total neutron population in the system at time t, is divided by the neutron loss rate.

The time rate of change of the neutron population is

$$dN/dt = P(t) - L(t)$$

$$= \text{production rate - loss rate.}$$

This can be rewritten as

$$\frac{dN}{dt} = \frac{(k-1)}{\ell} * N(t)$$

For the case in which both k and ℓ are time independent, the solution is

$$N(t) = N_0 \exp\left[\left(\frac{k-1}{\ell}\right)t\right]$$

The value of k determines the state of the system.

- if $k < 1$, N(t) decreases exponentially and the system is *subcritical*
- if $k = 1$, N(t) is constant, and the system is *critical*
- if $k > 1$, N(t) increases exponentially, and the system is *supercritical*

The neutron lifetime is typically in the range 10^{-3} to 10^{-4} seconds in a thermal multiplying system and 10^{-5} to 10^{-6} in a fast system. For a value of k = 1.001 and a lifetime of 10^{-4} seconds, the power level of this system would increase by e^{10} = 22,000 in one second, which would make it difficult to control.

Fortunately, there is a mitigating factor. A small fraction of the fission neutrons are not *prompt*, that is, emitted at the time of the fission event, but are *delayed*. This fraction, although small (0.6% for ^{235}U and 0.2% for ^{239}Pu and ^{233}U), is sufficient to allow control of systems for which the multiplication factor is less than prompt critical. *Prompt critical* is defined as the condition when $k = 1 + \beta$, where β is the *delayed neutron fraction*. As long as a system maintains k_{eff} less than prompt critical, the system can be controlled. Using the delayed neutron fractions given above, the prompt critical values of k_{eff} become 1.006 for ^{235}U and 1.002 for ^{239}Pu and ^{233}U. As close as these values are to 1.00, it behooves the criticality specialist to stay well below a value of $k_{eff} = 1.00$ in criticality safety analyses.

NEUTRON FATE

Consider the fate of fission neutrons - they are born, they can leak away or they can be absorbed. In an absorption, the neutron is lost, but it may cause another fission and produce more neutrons. Define the probabilities for each of these events.

P_{NL} = probability that a neutron will not leak out of system before absorption

P_{AF} = probability that if a neutron is absorbed it will be absorbed in the fissionable material

P_f = probability that if a neutron is absorbed in fissionable material, it will induce a fission reaction

First consider a thermal system and calculate the ratio of two interaction rates.

$$P_{AF} = \frac{\phi \Sigma_a^f}{\phi \Sigma_a} = \frac{\Sigma_a^f}{\Sigma_a} \equiv f$$

Where Σ_a is the total absorption cross section, Σ_a^f is the absorption cross section of the fissionable material and *f* is defined as the thermal utilization. Defining *f* this way implies that all of the important absorptions occur at thermal energies. Also implicit in this statement is that the absorption rate is everywhere proportional to the cross section, i.e., that the flux is uniform and cancels in the ratio of the interaction rates.

Similarly, P_f can be defined as the ratio of two interaction rates.

$$P_f = \frac{\phi \Sigma_f^f}{\phi \Sigma_a^r} = \frac{\Sigma_f^f}{\Sigma_a^f}$$

where Σ_f^f is the fission cross section of the fissionable material.

Starting with N_1 neutrons in one fission generation, then the number of neutrons in the next generation, N_2, is

$$N_2 = \nu P_f P_{AF} P_{NL} N_1$$

$$= \eta f P_{NL} N_1$$

where

$$\eta = \frac{\nu \Sigma_f^f}{\Sigma_a^f}$$

The multiplication factor becomes

$$k = \frac{N_2}{N_1} = \eta f P_{NL}$$

If the system is very large so that the nonleakage probability is unity then

$$k_\infty = \eta f = \frac{\nu \Sigma_f^f}{\Sigma_a}$$

This quantity, k_4, is called the *infinite multiplication factor* and is a characteristic of the materials only.

To this point we have only considered absorption of thermal neutrons. Now include "fast" and "slowing down" effects. Define the *fast fission factor* ε to be

$$\varepsilon \equiv \frac{\text{total number of fission neutrons from fast and thermal fissions}}{\text{number of fission neutrons from thermal fissions}}$$

Define the *resonance escape probability*, p, as

p = fraction of fission neutrons that slow down from fission to thermal energy without being absorbed.

Combining these parameters gives the *four-factor formula* for an infinite (large) system

$$k_4 = \eta f p \varepsilon$$

Typical values for these parameters are: ε from 1.00 to 1.02; p from 0.7 to 0.95; f = 0.7 to 0.9; and η = 2.0 to 2.2.

Now separate the nonleakage probability into fast, P_{FNL}, and thermal, P_{TNL}, components

$$P_{NL} = P_{FNL} P_{TNL}$$

to get the *six factor formula* for smaller systems

$$k = \eta f p \varepsilon P_{FNL} P_{TNL}$$

This approach seems very elementary; however, it is a useful way to understand what is going on in a fissioning system. If each of the six factors can be calculated, which may not be a trivial task, then it is known whether or not the system is critical. In criticality analysis, each of these terms is very rarely calculated; however, the criticality specialist should be aware of the processes of thermal and fast absorption, leakage, and neutron slowing down.

As an example of a quick hand calculation, consider the thermal-energy values for ^{233}U, ^{235}U and ^{239}Pu. (Reference: Neutron Cross Sections, Vol. 1, Neutron Resonance Parameters and Thermal Cross Sections, S. F. Mughabghab, Academic Press, Inc., New York, 1984.)

	^{233}U	^{235}U	^{239}Pu
ν	2.493	2.425	2.877
σ_a	574.7	680.9	1017.3
σ_f	529.1	582.6	748.1
k_4	2.296	2.074	2.115

These values indicate that the maximum multiplication factors possible in a ^{233}U, ^{235}U or ^{239}Pu system are 2.296, 2.074 and 2.115, respectively. One qualifier to this is that this calculation is done for the thermal neutron values only. The quantities in the table change with neutron energy and there are neutron spectrum effects which result in different values for real systems. However, the spectrum effects do not drastically change these values.

MEAN FREE PATH

So far this module has introduced the concepts of neutron multiplication and neutron lifetime. Another concept to consider is the neutron mean free path. The question is, how far will a neutron travel, on the average, from the location where it is born to the point at which it is captured? The *mean free path* is defined as the reciprocal of the macroscopic cross section. However, that definition presents some difficulties. As the neutron slows from fast energies to thermal energies, the energy dependent cross section varies. Even at thermal energies, where a single energy-independent cross section may be appropriate, several mean free paths could be calculated: mean free path for absorption, mean free path for scattering, etc. A mean free path for scattering only gives an indication of the distance the neutron travels before a scattering event, but the neutron may have several consecutive scattering events and travel further than one mean free path.

To give some idea of the distances involved consider a few cases and approximate values for mean free paths.

System Type	Neutron Energy	Neutron Event	Mean Free Path (cm)
Uranium Metal	2 MeV	Interaction	~ 3
Water	Thermal	Interaction	~ 0.66
Water	Thermal	Absorption	~ 45
Water, 30 g/L ^{239}Pu	Thermal	Absorption	~ 13
Water, 300 g/L ^{239}Pu	Thermal	Absorption	~ 1.3

In general, for many mixtures of interest to the criticality specialist the mean free path for a neutron is on the order of a few centimeters.

Related to the neutron mean free path is the size of the fissionable volume. It is apparent that if the material volume is large most of the neutrons will interact in the active material and the neutron leakage fraction will be small. Conversely, a small system may have a large leakage fraction. The *mean chord length*, defined as 4V/S, (V/S is the volume divided by the surface area) is a useful parameter to estimate whether a system is large or small. If the mean chord length is large compared to the neutron mean free path, it might be expected that the leakage fraction is relatively small. The mean chord lengths for some geometries are listed in the table below.

Geometry	Volume	Surface Area	Mean Chord Length
Long Cylinder (negligible end areas) radius r arbitrary height h	$\pi r^2 h$	$2\pi rh$	2r
Large Slab thickness t arbitrary face area A	At	2A	2t
Sphere radius r	$(4/3)\pi r^3$	$4\pi r^2$	(4/3) r
Cylinder with L/D = 1 radius r	$2\pi r^3$	$6\pi r^2$	(4/3) r
Cube length of side x	x^3	$6x^2$	(2/3) x

As a general rule-of-thumb if the mean chord length is a few neutron mean free path lengths, the system is "leaky" and if the mean chord length is many neutron mean free path lengths the leakage is small.

SUMMARY

In this module, the basic concepts and parameters of fission chain reactions were introduced, including the definitions of infinite multiplication factor, effective multiplication factor, subcritical, critical, supercritical, neutron production and loss, resonance escape probability, four factor and six factor formulae, mean free path and mean chord length.

The effect of some of these parameters on the multiplication of the system were discussed and the use of mean free path and mean chord length to to get an idea of the "leakiness" of a system was illustrated.

PROBLEMS

1. Compute and plot the ratio $\nu\Sigma_f/\Sigma_a$ for a mixture of ^{235}U and ^{238}U metal as a function of the uranium enrichment (from 0.7 % to 100 %) for thermal energy cross sections with these values. Assume the density of uranium is 18.9 g/cm^3.

	^{235}U	^{238}U
ν (n per fission)	2.425	
σ_a (barns)	680.9	2.68
σ_f (barns)	582.6	
Atomic Weight	235.0439	238.0508

2. A mixture of ^{235}U and water is contained in a sphere with radius 14.20 cm. The atom densities and thermal neutron cross sections are listed in this table.

	at/b-cm	σ_f	σ_a	σ_s
^{235}U	1.2811E-04	582.6	680.9	14.3
H	6.6559E-02		0.333	20.49
O	3.3279E-02		0.00054	3.76

Estimate the mean free path for a thermal neutron in this mixture and compare it with the mean chord length in the sphere.

3. Calculate the ratio of the hydrogen atom density to the fissile atom density, N_H/N_F, for a critical ($k_4 = 1.0$) infinite homogenous mixture of the fissile material and water. Do calculations for ^{235}U, ^{233}U, and ^{239}Pu separately. For ^{235}U repeat the calculation with 5 atom-% ^{238}U in the uranium and with 93 atom-% ^{238}U. Repeat the plutonium calculation with 5 atom-% ^{240}Pu in the plutonium mixture. Use the following thermal neutron values.

	ν	σ_a	σ_f
^{235}U	2.425	680.9	582.6
^{239}Pu	2.877	1017.3	748.1
^{233}U	2.493	574.7	529.1
H		0.333	
O		0.00054	
^{238}U		2.68	
^{240}Pu		289.5	

4. In this module an equation was developed for the time rate of change for the neutron population. This equation is

$$N(t) = N_0 \exp\left[\left(\frac{k-1}{\ell}\right)t\right]$$

The equation does not include effects of delayed neutrons, which are important to the control of a chain reaction. An approximation in which the delayed neutrons are assumed to be emitted in a single group with an average decay constant gives the following result for the time rate of change of the neutron population

$$N(t) = N_0\left[\left(\frac{\beta}{\beta-\rho}\right)\exp\left(\frac{\lambda\rho}{\beta-\rho}\right)t - \left(\frac{\rho}{\beta-\rho}\right)\exp\left(-\frac{\beta-\rho}{\ell}\right)t\right]$$

where β is the delayed neutron fraction, ρ, is the reactivity ($\rho = (k-1)/k$), λ is the decay constant for the single delayed neutron group, and ℓ is the mean lifetime of a neutron from birth in fission to capture or escape. Typical values for a ^{235}U thermal system are $\beta = 0.0075$, $\lambda = 0.08$ sec^{-1}, and $\ell = 0.001$ sec. For a system which starts from critical the reactivity ρ . k-1. Calculate the time rate of change of the neutron population for values of ρ between 0.0 and β.

PROBLEM SOLUTIONS

1. The macroscopic cross section is defined as the product of the atom density and the microscopic cross section (see NCSET Module 1). Given the enrichment of the uranium in wt-%, the atom densities of the two uranium isotopes are:

$$N_{235} = (N_a / A_{235}) * \rho * (\text{wt-\%} / 100)$$

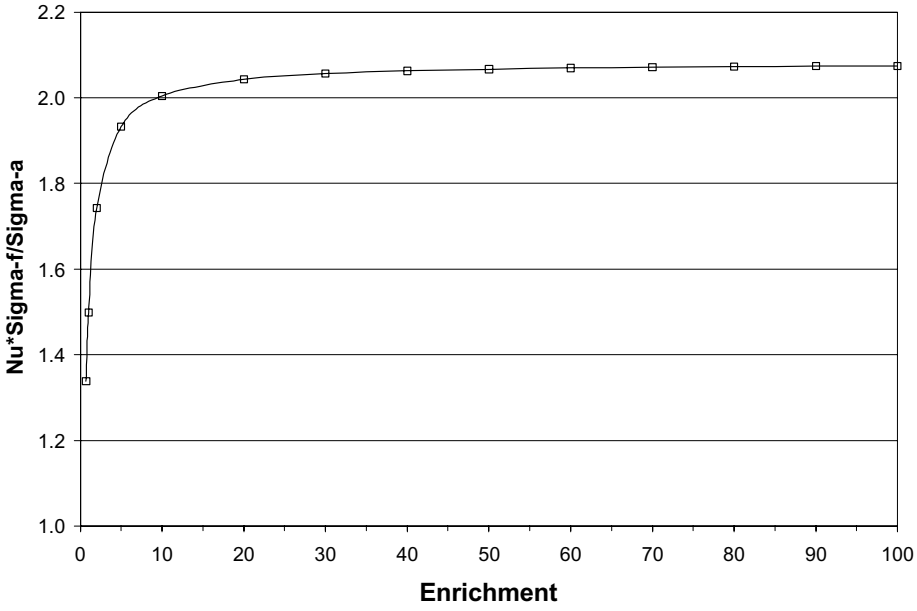
$$N_{238} = (N_a / A_{238}) * \rho * (1-(\text{wt-\%} / 100))$$

.

Using these equations with $N_a = 0.60221$ plus the data given in the problem, the following table can be generated. Note that the total macroscopic absorption cross section is the sum of the values for the two isotopes.

Enrichment	N_{235}	N_{238}	Σ_f	Σ_a	$\nu*\Sigma_f/\Sigma_a$
0.7	3.390e-04	4.748e-02	0.197	0.358	1.338
1	4.842e-04	4.733e-02	0.282	0.457	1.498
2	9.685e-04	4.686e-02	0.564	0.785	1.743
5	2.421e-03	4.542e-02	1.411	1.770	1.932
10	4.842e-03	4.303e-02	2.821	3.413	2.005
20	9.685e-03	3.825e-02	5.642	6.697	2.043
30	1.453e-02	3.347e-02	8.464	9.981	2.056
40	1.937e-02	2.869e-02	11.285	13.266	2.063
50	2.421e-02	2.391e-02	14.106	16.550	2.067
60	2.905e-02	1.912e-02	16.927	19.834	2.070
70	3.390e-02	1.434e-02	19.748	23.119	2.071
80	3.874e-02	9.562e-03	22.569	26.403	2.073
90	4.358e-02	4.781e-03	25.391	29.688	2.074
100	4.842e-02	0.000e+00	28.212	32.972	2.075

The following figure shows a graphical representation of these results.



This points to the importance of ^{238}U but is not a realistic calculation because it does not include the resonance absorption in ^{238}U which is significant in this mixture.

2. From Page 6, the mean free path is the reciprocal of the total macroscopic cross section. For a mixture of materials and interactions, the total cross section is the sum of the individual components (see NCSET Module 1). This gives

$$\Sigma_{TOT} = N_{235} * (\sigma_f + \sigma_a + \sigma_s)_U + N_H * (\sigma_a + \sigma_s)_H + N_O * (\sigma_a + \sigma_s)_O$$

Inserting the numbers from the problem,

$$\Sigma_{TOT} = 0.1637 + 1.3860 + 0.1251 = 1.6748 \text{ cm}^{-1}$$

The inverse of this is the mean free path, 0.5971 cm.

For a sphere the mean chord length (see Page 7) is given by (4r/3). With a radius 14.2 cm the mean chord length is 18.93 cm. For this case the mean chord length is much larger than the mean free path, indicating that the leakage is probably not significant for this sphere.

3. From Page 4, the infinite multiplication factor is

$$k_\infty = \frac{\nu\Sigma_f}{\Sigma_a} = \frac{\nu N_F \sigma_f}{N_F\sigma_a^F + N_H\sigma_a^H + N_O\sigma_a^O + N_D\sigma_a^D}$$

where the subscripts on the atom densities are F for fissile atoms, H for hydrogen, O for oxygen, and D for ^{238}U or ^{240}Pu. The subscripts on σ are a for absorption and f for fission, and the superscripts are F for fissile atoms, H for hydrogen, O for oxygen and D for ^{238}U or ^{240}Pu. Since the system is critical, $k_4 = 1.0$ and $N_O = 0.5\ N_H$ (i.e., H_2O); with a little algebraic manipulation the equation can be rearranged to

$$\frac{N_H}{N_F} = \frac{\nu\sigma_f - \sigma_a^F - \sigma_a^D\left(\frac{N_D}{N_F}\right)}{\sigma_a^H + 0.5\,\sigma_a^O}$$

Since the atom-fraction of diluent atoms (^{238}U or ^{240}Pu) is just N_D/N and the atom-fraction of fissile atoms is N_F/N, the ratio of N_D/N_F is just the ratio of the given atom-fractions. For example, for the 5 at-% ^{238}U case, $N_D/N_F = 0.05/0.95$.

The results are tabulated below. Note that the inclusion of ^{238}U in the uranium mixture or ^{240}Pu in the plutonium mixture does not significantly change the result.

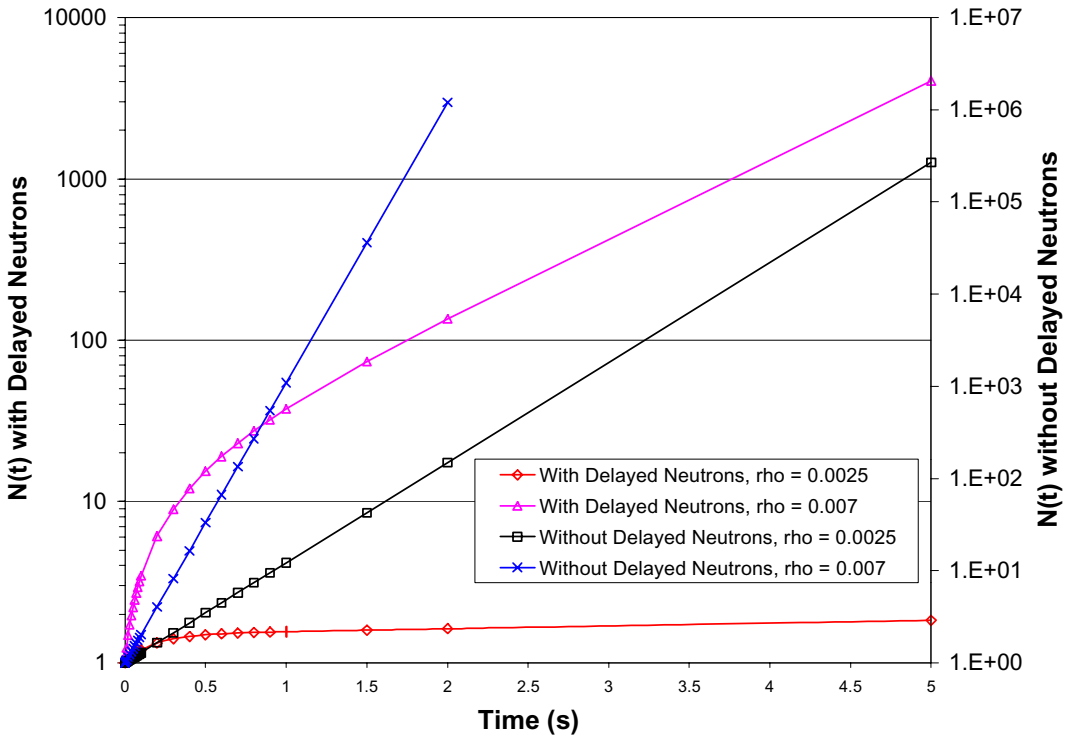
Material	N_H / N_F
^{235}U	2196
^{239}Pu	3406
^{233}U	2233
^{235}U + 5 at-% ^{238}U	2196
^{235}U + 93 at-% ^{238}U	2089
^{239}Pu + 5 at-% ^{240}Pu	3360

4. The change in the neutron population as a function of time is calculated by simply substituting the values of the variables into the equations and letting t vary from a fraction of a second up to a few seconds. The following table shows the calculated neutron population as a function of time for rho equal to 0.0025 and equal to 0.007, both with and without the delayed neutrons. For the first case and with the approximation of a single delayed neutron group, the neutron population doubles in about five seconds. Without the delayed neutrons the $\rho = 0.0025$ calculation indicates that the population would increase by a factor of over 200,000 in the same time interval. Note that as rho approaches β in the $\rho = 0.007$ columns, the increase in neutrons is incredibly fast, reaching 4000 times the initial value in five seconds with the delayed neutrons and increasing by more than 10^{15} with no delayed neutrons. In a more usual transient analysis about six groups of

delayed neutrons would be used to better approximate reality. Each of the six groups would have different values of β and λ.

	rho = 0.0025		rho = 0.007	
t	N(t) with Delayed n	N(t) without Delayed n	N(t) with Delayed n	N(t) without Delayed n
0.0001	1.0003	1.0003	1.0024	1.0007
0.001	1.0026	1.0025	1.0238	1.0070
0.005	1.0126	1.0126	1.1192	1.0356
0.01	1.0250	1.0253	1.2388	1.0725
0.02	1.0488	1.0513	1.4791	1.1503
0.03	1.0714	1.0779	1.7210	1.2337
0.04	1.0930	1.1052	1.9645	1.3231
0.05	1.1136	1.1331	2.2096	1.4191
0.06	1.1332	1.1618	2.4564	1.5220
0.07	1.1519	1.1912	2.7049	1.6323
0.08	1.1696	1.2214	2.9550	1.7507
0.09	1.1866	1.2523	3.2069	1.8776
0.1	1.2027	1.2840	3.4605	2.0138
0.2	1.3281	1.6487	6.0983	4.0552
0.3	1.4065	2.1170	8.9402	8.1662
0.4	1.4565	2.7183	12.0155	16.4447
0.5	1.4893	3.4903	15.3569	33.1155
0.6	1.5115	4.4817	19.0008	66.6863
0.7	1.5275	5.7546	22.9876	134.2898
0.8	1.5396	7.3891	27.3623	270.4264
0.9	1.5494	9.4877	32.1749	544.5719
1	1.5578	12.1825	37.4814	1096.633
1.5	1.5925	42.5211	73.8702	3.63e+04
2	1.6249	148.4132	135.7497	1.20e+06
5	1.8321	2.68e+05	4.06e+03	1.59e+15
10	2.2377	7.20e+10	1.10e+06	2.52e+30

The following figure graphically presents these results. Note the logarithmic scale on the plot.


10000
1000
100
10
1
1.E+07
1.E+06
1.E+05
1.E+04
1.E+03
1.E+02
1.E+01
1.E+00
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1
1.5
2
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3
3.5
4
4.5
5
N(t) with Delayed Neutrons
N(t) without Delayed Neutrons
Time (s)
With Delayed Neutrons, rho = 0.0025
With Delayed Neutrons, rho = 0.007
Without Delayed Neutrons, rho = 0.0025
Without Delayed Neutrons, rho = 0.007